AUTODESK UNIVERSITY

ES125256

Autodesk Nastran for Inventor: Unlocking Dynamics

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Learning Objectives

- Learn how to recognize the characteristics of dynamic response and other advanced product behaviors
- Learn how to simplify CAD for vibration analyses
- Discover loads and constraints for vibration analyses
- Learn how to interpret results of vibration simulation

Description

As success and comfort levels increase, engineers need to look beyond the limited questions that can be asked of this linear static solution to truly understand and respond to the demands of innovation. Autodesk Nastran In-CAD software brings advanced analysis techniques to the Inventor platform, letting engineers explore nonlinear, vibratory buckling and other behaviors or failure conditions within the context of an interactive design and modeling session. You will learn to recognize the need for these solutions and how to take your Inventor digital prototypes through advanced tests. Additionally, you will be introduced to the inputs required and the outputs produced so more-exacting and more-precise engineering decisions can be made.

Speakers

Tony Abbey
Tony Abbey has created and taught a wealth of FEA training material over the past 20 years. Thousands of engineers across the world have benefited from his live and e-learning based classes. He has developed a reputation for providing the student with an experience that is full of insight gained from his extensive experience, but which also challenges and motivates. Tony has been working with FEA for nearly 40 years, both in Industry and for leading FEA software providers in the UK and the US. His informal and interactive presentation style allows the key concepts to be taught in a manner which involves participants fully in the course material. Tony presents the popular NAFEMS live e-Learning classes, covering a wide FEA subject area. He writes a series of authoritative articles on FEA for Digital Engineering magazine each month.

Mitch Muncy
Mitch is a Product Manager responsible for the Mechanical FEA strategy at Autodesk. This includes overseeing products such as Autodesk Simulation Mechanical, Autodesk Nastran, and Autodesk Nastran In-CAD, as well as the Mechanical FEA roadmap on Forge.

Introduction to Dynamics

Before examining dynamics in detail, it is worth reviewing exactly what dynamics are. Dynamics can be thought of as the kinetic behavior of an object. That is, if an object is moving, it is involved in a dynamic activity. However, from an analytical point of view, moving objects can usually be divided into two classes: Those where the object is moving through space as a rigid body (called kinematics) and those where the body itself is deforming (what we call dynamics). Further, the relative magnitude of the motion or deformation needs to be considered as well. If the object is moving very slowly, it could still be considered a static problem for any arbitrarily short period of time. A further refinement to the definition of dynamics would relate to the loading and motion.

If the rate of loading is such that it is roughly equal to the object's natural frequency, then the response will be dynamic.

That is, the loading will interact with the system's natural frequencies. At slow loading rates, the natural frequencies are not excited and the model can be considered static. In the dynamic range, the loading will excite certain natural frequencies which will change the response from a strictly static sense. At higher frequencies, the system will not have time to respond to the excitation, and the excitation will have little effect on the system.

Types of Dynamics Problems

There are several types of problems that are analyzed with finite elements that are dynamic in nature.

Eigenvalue Analysis

Most common are a system's natural frequencies. Any non-rigid system will have one or more natural vibration frequencies. And Eigenvalue analysis will find these frequencies and the corresponding Eigenvectors or mode shapes.

Harmonic Response

If a system is known to be dynamic, a harmonic analysis will calculate the response of the system to a series of enforced sinusoidal loads. If the load is assumed to continue indefinitely, the solution to these problems can be found in closed form, resulting in a series of static-like results at a series of excitation frequencies.

Transient Response

If the loading in not periodic, it may be necessary to load the model in the time domain. In this case, the model is solved at a series of time steps that trace the response of the system over time. A transient analysis does this, using the solution at each time step as the initial condition for the next time step.

Random Response

If the actual excitation of a structure is unknown, but the loading can be roughly quantified as a power spectrum, a random analysis can be done. A random analysis is run after a frequency response analysis, and the responses at the different frequencies are combined into a single result based on the relative magnitude of the spectrum at

different frequencies. The result is a single set of results, incorporating the contributions of all the different excitation values.

Response Spectrum Analysis

A response spectrum analysis is similar to a random analysis in that the actual loads are unknown. In a spectral response solution, the peak accelerations at different frequencies are specified instead of a power density. But like the random analysis, the results at different natural frequencies are combined to produce a single result.

Eigenvalue Analysis – Normal Modes or Natural Frequency

Eigenvalue analysis forms the basis of several other dynamic analyses, including the modal approach formulations and response spectrum analysis. In addition, the results of a modal analysis will tell you much about the dynamic characteristics of the model, including whether or not you really have a dynamic system at all. For this reason, analysts often run an Eigenvalue analysis first to examine the basic model dynamics and to check for modeling problems.

Eigenvalue Theory

Consider a dynamic system. In general the equations of motion can be expressed as a function of the system mass [M], stiffness [K], damping [B] and applied loads [P]:

$$[M]\{\ddot{x}(t)\}+[B]\{\dot{x}(t)\}+[K]\{x(t)\}=\{P(t)\}$$

Eigenvalues or natural frequencies are found when there is no damping or applied loads. The equations of motion for free vibration can then be written as:

$$[M]\{\ddot{x}(t)\}+[K]\{x(t)\}=0$$

If we assume a sinusoidal vibration where the displacement can be described by $\{x(t)\}=\{\phi\}e^{i\omega t}$, we can replace the $\{x(t)\}$ term with this. In addition, for a sinusoidal variation, the acceleration is the second derivative of the displacement, $\{x(t)\} = -\omega^2\{\phi\}e^{i\omega t}$. The equation is then:

$$(-\omega^2[M]+[K])[\phi]e^{i\omega t}=0$$

$e^{i\omega t}$ is never zero, so the equation can be rearranged to the form of a general Eigenvalue problem. Autodesk Nastran determines natural frequency by solving the Eigenvalue problem:

$$\left|[K]-\lambda[M]\right|[\phi]=0$$

where,

[K] is the global linear stiffness matrix

[M] is the global mass matrix

λ is the Eigenvalue for each mode that yields the natural frequency = ω^2

ϕ is the Eigenvector for each mode that represents the natural mode shape

The Eigenvalue is related to the system's natural frequency

$$\lambda_i = \omega_i^2$$

ω is the circular frequency (radians per second)

Or in hertz:

$$\omega = 2\pi f$$

f is the cyclic frequency (hertz)

One solution is trivial (ϕ =0), but the other solutions for ω are interesting. ω^2 is called the Eigenvalue λ, and each λ is accompanied by a unique $\{\phi\}$ called the Eigenvector.

In solving the above Eigenvalue problem there are as many Eigenvalues and corresponding eigenvectors as there are unconstrained degrees of freedom. Often, however, only the lowest natural frequencies are of practical interest. The lowest frequency mode will always be the first mode extracted.

The solution of the Eigenvalue problem is difficult and a number of different approaches have been developed over the years. Currently the Lanczos approach is favored as it is fast, accurate and robust.

Also, while the λ found is the exact Eigenvalue, the Eigenvectors are arbitrarily scaled. That is, there is no unique magnitude to the vectors. They simply represent a shape. By default Autodesk Nastran performs a mass scaling on the vectors. This is done by calculating the generalized mass of the model from the equation:

$$M = [\phi]^T [M][\phi]$$

All of the terms of the vector are then divided by it. This results in a seemingly arbitrary scaling of the vectors, but it has important mathematical properties that can be exploited elsewhere. In addition to mass scaling, Nastran also has max scaling available, where the largest value in the vector will be 1.0. This allows small vectors to be examined manually.

A property of Eigenvectors is that they are orthogonal. This means that one Eigenvector multiplied by another will produce an identity matrix. An Eigenvector vector multiplied by itself will be zero.

$$[\phi]_i [\phi]_i = [0]$$
$$[\phi]_i [\phi]_j = [I]$$

This is another property that is exploited in dynamics solutions.

Eigenvalue analyses are usually divided into two types based on the constraints. If the model is unconstrained, it is referred to as a free-free analysis, after the corresponding beam representation i.e. a beam analyzed this way would be free at both ends. When an Eigenvalue analysis is run on a free-free system, there will be six zero-frequency rigid body modes found in addition to the elastic modes. These modes represent the free translation and rotation of the system in the six directions of motion. It is always a good model check to run an unconstrained modal analysis to assure that you are finding these modes as it assures that the model is not internally constrained accidentally. Some systems really are unconstrained, and unlike a static analysis, a modal analysis can be run successfully on a free-free structure.

The other type of modal analysis is a constrained system. In this case there should be no zero-frequency modes. IF any are found, it is an indication that some portion of the model is free to move in a rigid body manner. This type of motion is usually referred to as a mechanism. And while a modal analysis can solve this type of problem, a static analysis will fail. For this reason, a modal analysis is often used as a debugging tool for a static analysis that has failed. Once the source of the zero-frequency mode is identified, it can be constrained and a static analysis successfully run.

Direct Solution Techniques

Direct Transient Response

Transient analysis calculates the response of a system to a load over time. The load applied to the system can vary over time or simply be an initial condition that is allowed to evolve over time.

Direct Transient Response Theory

In direct transient response structural response is computed by solving a set of coupled equations using direct numerical integration. The method used is the same as for nonlinear transient response and allows for an adaptive time stepping algorithm. We begin with the dynamic equation of motion in matrix form:

$$[M]\ddot{x}(t) + [B]\dot{x}(t) + [K]x(t) = \{P\}(t)$$

The fundamental structural response (displacement) is solved at discrete times, typically at a series of time steps with a constant time increment between them The solution strategy is called the central finite difference method and is based on finding the displacement, velocity and acceleration at subsequent times, knowing those values at the current and past times.

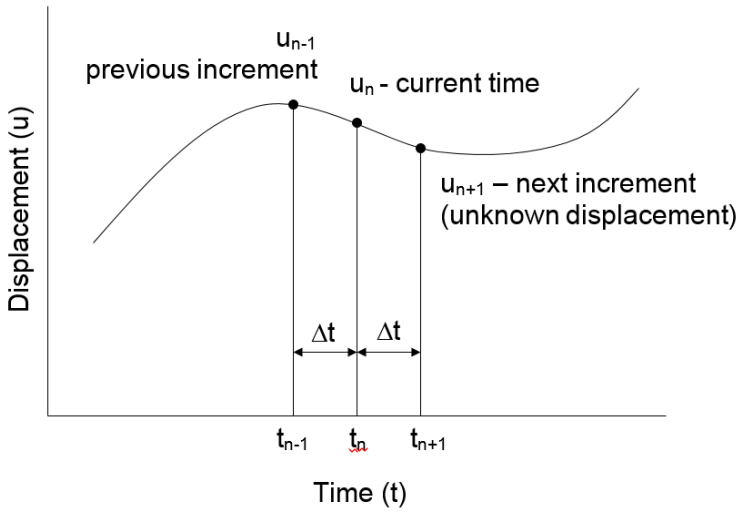

FIGURE 1: TRANSIENT INTEGRATION

At any time n, we know the displacement, velocity and acceleration. Using this information, there are a number of ways to find the displacement, velocity and acceleration at a subsequent step. For example, if we assume the current velocity remains constant until the next time increment, the subsequent displacement u_{n+1} would simply be the current location plus the velocity times the time increment, $u_n + V\Delta t$. This is true if we assume the current acceleration remains constant to the next increment. However, this cannot necessarily be assumed in a general problem. The mean value theorem posits an acceleration A_γ is a function of a constant gamma such that that acceleration is a weighted average of the accelerations and n and n+1

$$A_\gamma = \gamma A_n + (1-\gamma)A_{n+1}$$

Newmark was able to show that 0.5 is a good value for gamma, such that

$$A_\lambda = \frac{1}{2}(A_n + A_{n+1})$$

And

$$V_{n+1} = V_n + \frac{1}{2}\Delta t(A_n + A_{n+1})$$

Since velocity is the first derivative of displacement, and acceleration the second, they are usually represented in dot notation as their derivatives.

$$\dot{u}_{n+1} = \dot{u}_n + \frac{1}{2}\Delta t(\ddot{u}_n + \ddot{u}_{n+1})$$

However, since the acceleration may vary with time as well, we can propose that the displacement will be corrected by both velocity and acceleration terms thus:

$$u_{n+1} = u_n + \Delta t\dot{u}_n + \frac{1}{2}\Delta t^2\ddot{u}_\beta$$

The acceleration u_β in the above term is calculated like the velocity above, such that

$$\ddot{u}_\beta = (1-2\beta)\ddot{u}_n + 2\beta\ddot{u}_{n+1}$$

U can then be written in terms of the beta constant:

$$u_{n+1} = u_n + \Delta t\dot{u}_n + \frac{1-2\beta}{2}\Delta t^2\ddot{u}_n + \beta\Delta t^2\ddot{u}_{n+1}$$

The beta value in the above equation is called the Newmark beta, and can vary between 0 and 1. It is usually used as ¼, which yields the constant average acceleration method, where

$$u_{n+1} = u_n + \Delta t \dot{u}_n + \frac{1}{4}\Delta t^2(\ddot{u}_n + \ddot{u}_{n+1})$$

These equations can be manipulated to find expressions for velocity and acceleration at the current time increment, but expressed in terms of the displacement at past and subsequent time intervals.

$$\{\dot{u}_n\} = \frac{1}{2\Delta t}\{u_{n+1} - u_{n-1}\}$$

$$\{\ddot{u}_n\} = \frac{1}{\Delta t^2}\{u_{n+1} - 2u_n + u_{n-1}\}$$

These representations are then substituted into the equations of motion, resulting in the following.

$$\left[\frac{m}{\Delta t^2}\right](u_{n+1} - 2u_n + u_{n-1}) + \left[\frac{b}{2\Delta t}\right](u_{n+1} - u_{n-1}) +$$

$$\left[\frac{k}{3}\right](u_{n+1} + u_n + u_{n-1}) = \frac{1}{3}(P_{n+1} + P_n + P_{n-1})$$

However, since u_{n+1} is the unknown, we need a u_{n+1} value in each expression for the solution. We will represent the displacement at the current time as the average over the adjacent times. Likewise the load can be averaged over three steps, the difference being that we know P_{n+1} as it is an input value.

Using these average values:

$$u_n = \frac{(u_{n-1} + u_n + u_{n+1})}{3}$$

$$P_n = \frac{(P_{n-1} + P_n + P_{n+1})}{3}$$

We then rearrange the terms to end up with the unknown u_{n+1} on the left side and the known u_n and u_{n-1} on the right:

$$[A_1]\{u_{n+1}\} = [A_2] + [A_3]\{u_n\} + [A_4]\{u_{n-1}\}$$

$$\text{where } [A_1] = \left[\frac{M}{\Delta t^2} + \frac{B}{2\Delta t} + \frac{K}{3}\right]$$

$$[A_2] = \frac{1}{3}\{P_{n+1} + P_n + P_{n-1}\}$$

$$\left.\begin{aligned}[A_3] &= \left[\frac{2M}{\Delta t^2} - \frac{K}{3}\right] \\ [A_3] &= \left[\frac{-M}{\Delta t^2} + \frac{B}{2\Delta t} - \frac{K}{3}\right]\end{aligned}\right\}$$

To solve this equation, we need to calculate the four A terms, then decompose (invert) the A_1 term to find u_{n+1}. It should be noted that for a problem with a constant time step that A_1 need be decomposed only once. However, if the time step changes, it will be necessary to do the calculation again. Thus, it is advisable to maintain a constant time step unless changing it will offset the extra cost.

Direct Frequency Response

Direct Frequency Response Theory

A direct frequency response starts with the general equations of motion, but assumes an oscillating load

$$[M]\ddot{x}(t) + [B]\dot{x}(t) + [K]x(t) = \{P(\omega)\}e^{i\omega t}$$

We can them propose that the solution is also in the form of an oscillating function

$$x(t) = \{u(\omega)\}e^{i\omega t}$$

U(ω) is a complex displacement vector. The velocity and acceleration can be found by taking the derivative.

$$\dot{x}(t) = i\omega\{u(\omega)\}e^{i\omega t}$$

$$\ddot{x}(t) = -\omega^2\{u(\omega)\}e^{i\omega t}$$

Substitute this into the equation of motion and divide by the $e^{i\omega t}$ term to get

$$[-\omega^2 M + i\omega B + K]\{u(\omega)\} = \{P(\omega)\}$$

The frequency w is a constant in this equation. Therefore, the solution will yield a complex displacement vector u for each frequency that is selected.

In a direct frequency response analysis, this equation is solved repeatedly for each selected frequency. As a result, the solution time is proportional to the number of frequencies that are selected for solution.

Modal Techniques

For large models, there is an alternate technique available for dynamics that reduces the solution degrees of freedom, and can significantly impact the run time. This alternate modal approach replaces the physical degrees of freedom with a reduced number of modal degrees of freedom. Fewer degrees of freedom mean a faster solution. This can be a big timesaver for transient models with a large number of time steps, and for frequency response models with large numbers of frequencies.

Modal Transient

To run a modal transient, it is necessary to transform the physical coordinates to modal coordinates. The natural frequencies and Eigenvectors are a good way to do this because of their property of orthogonality. As such, we can replace the physical coordinates u with the modal coordinates:

$$\{x(t)\} = [\phi]\{\xi(t)\}$$

The basic equation of motion (temporarily ignoring the damping term) becomes:

$$[M][\phi]\ddot{\xi}(t) + [K][\phi]\xi(t) = \{P(t)\}$$

With a little manipulation, we can rearrange this into something more useful:

$$[\phi]^T[M][\phi]\ddot{\xi}(t) + [\phi]^T[K][\phi]\xi(t) = [\phi]^T\{P(t)\}$$

But the mass and stiffness terms are now the generalized modal matrices, diagonal matrices that are easily handled:

$$[\phi]^T[M][\phi] = \text{Generalized Mass m}$$

$$[\phi]^T[K][\phi] = \text{Generalized Stiffness k}$$

The diagonal matrices have the effect of uncoupling the modal degrees of freedom. The load term is a vector and is already uncoupled. As a result, the system is easily solved as a series of uncoupled equations:

$$m_i\ddot{\xi}(t) + k_i\xi(t) = p_i(t)$$

m and k are the generalized mass and stiffness values for each modal degree of freedom, and p is the modal load vector.

Once values are found for the modal displacements, the physical displacements can be found from the sum of the modal displacements:

$$\{x(t)\} = [\phi]\{\xi(t)\}$$

This approach will yield the exact same answer as the direct approach, provided that all modal DOF are included in the transformation. However, the strength of the approach comes about because an answer that is very close to exact can usually be obtained with significantly fewer modal degrees of freedom than there are physical degrees of freedom. With fewer DOF, the solution can proceed much faster. This can be especially efficient for large models and for models that require many time steps.

Modal Frequency Response

Modal frequency response is implemented similarly to the modal transient. First a transformation is defined:

$$\{x\} = [\phi]\{\xi(\omega)\}e^{i\omega t}$$

This is substituted into the equations of motion:

$$-\omega^2[M]\{x\} + [K]\{x\} = \{P(\omega)\}$$

Resulting in the following:

$$-\omega^2[M][\phi]\{\xi(\omega)\} + [K][\phi]\{\xi(\omega)\} = \{P(\omega)\}$$

As with the transient case, pre-multiply by $[\phi]^T$:

$$-\omega^2[\phi]^T[M][\phi]\{\xi(\omega)\} + [\phi]^T[K][\phi]\{\xi(\omega)\} = [\phi]^T\{P(\omega)\}$$

As before, these terms are replaced with the uncoupled generalized components:

$[\phi]^T[M][\phi]$ = Generalized mass m

$[\phi]^T[K][\phi]$ = Generalized stiffness k

$$[\phi]^T\{P(\omega)\} = \text{Modal load vector p}$$

Resulting in an uncoupled series of equations that are easily solved:

$$-\omega^2 m_i\{\xi(\omega)\} + k_i\{\xi(\omega)\} = \{P_i(\omega)\}$$

And once the modal displacements ξ are found, the physical displacements can be frond from the sum of the modal displacements.

$$\{x\} = [\phi]\{\xi(\omega)\}e^{i\omega t}$$

Damping

So far we have generally ignored damping. However, damping is an important component of reality, and various types make up the [B] matrix in dynamic solutions.

The general damping equation used in Autodesk Nastran is as follows:

$$[B] = CB1*\left([B^1] + \frac{iG}{W_3}[K] + \frac{i1}{W_4}\sum G_E[K_E]\right) + CB2*[B^2] + \alpha[M] + \beta[K]$$

The various terms represent different kinds of damping that can be included in Nastran:

[B1] = Damping elements – CDAMP, CVIS and direct input B2GG matrices

CB1 = scale factor on damping elements

[B2] = B2PP direct input matrix

CB2 = scale factor on direct input damping

G = overall structural damping coefficient (stiffness dependent damping) - PARAM,G

W_3 = frequency of interest for overall structural damping – PARAM,W3 – for conversion of structural damping to equivalent viscous damping

[K] = global stiffness matrix

G_e = structural damping coefficients for different materials (on MATi cards)

W_4 = frequency of interest for element-based structural damping – PARAM,W4 – for conversion of structural damping to equivalent viscous damping

$[K_E]$ = Element stiffness matrices for elements with G_e specified for their material

α = mass proportional damping constant for Rayleigh damping

β = stiffness proportional damping constant for Rayleigh damping

Damping in Direct Transient Analysis

Structural damping is really a complex function of the stiffness matrix and is often called 'stiffness-proportional' damping:

$$[B] = iG[K]$$

When this is put into the equation of motion, the velocity term disappears:

$$[M]\ddot{x}(t) + (1 + iG)[K]x(t) = \{P\}(t)$$

However, the direct transient solution cannot include a complex term. Therefore, structural damping is converted to an equivalent viscous damping term instead. For a single DOF oscillator, the viscous damping will equal the structural damping at one frequency.

$$b\omega = Gk$$

If this relation is extended to multi DOF systems, the equivalent damping matrix [B] can be represented as a function of the frequency and the damping coefficient

$$[B] = \frac{G}{\omega}[K]$$

However, since ω is a constant, this exact relation is only true at a single frequency. As the frequency decreases, the damping will become smaller, and as the frequency increases, the damping will grow larger. Thus structural damping in a direct transient solution will tend to filter out high-frequency responses. Graphically, this looks like this:

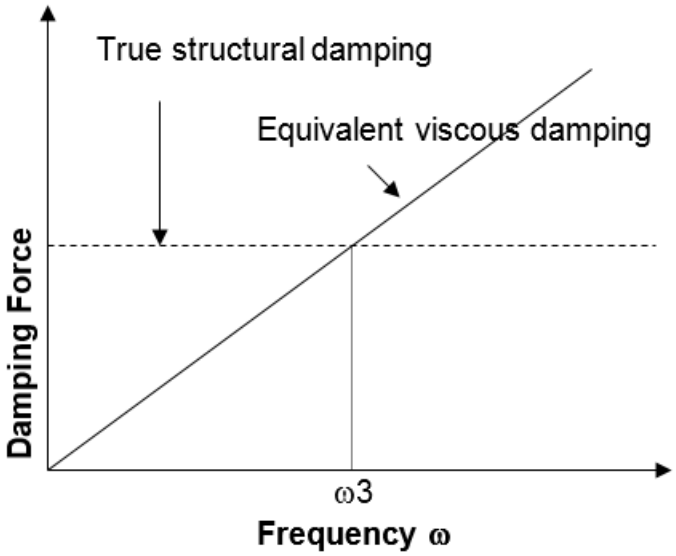

FIGURE 2: DAMPING FORCE – FREQUENCY

At frequency $\omega 3$, the equivalent viscous damping exactly equals the structural damping. But at lower and higher frequencies, the equivalent damping is not the same.

Damping in Modal Transient

Our solution to the modal transient problem relied on the mass and stiffness being uncoupled. That is, the $[\phi]^T[M][\phi]$ term was a diagonal matrix. We also ignored the damping for the time being. The reason for that is that the damping matrix rarely reduces out that way. $[\phi]^T[B][\phi]$ is not diagonal.

As a result of this, the modal approach in the presence of damping will revert to the direct solution, but in terms of modal coordinates. While this is still faster than using all the physical DOF, it is not as fast as solving the uncoupled equations.

To utilize the faster uncoupled equations, it is possible to create a diagonal damping matrix. This is done by applying damping individually to the modal DOF instead of constructing a physical damping matrix and transforming it modal coordinated. When this is done, the damping matrix is diagonal, and the equations are uncoupled, leading to much faster solution times.

Damping in Direct Frequency Response

In a direct frequency response problem, you are using a complex solution. As a result, the complex damping term can be used as is. It is not necessary to convert the structural damping to equivalent viscous damping. All forms of damping can be used without penalty.

Damping in Modal Frequency Response

In the modal solution, you run into the same issue with the transient, in that the damping matrix does not generally diagonalize. Further, any structural damping will create a complex stiffness matrix that does not diagonalize either. As a result, if either of those types of damping are included, the solution must be solved as in the direct method, but with modal coordinates. As with the transient case, it is faster than solving in physical space, but not so fast as solving the uncoupled equations. In order to reap the benefits of the uncoupled solution, it is necessary to forgo the other kinds of damping and use only modal damping, which damps individual modal DOF individually.

Non-Deterministic Analyses

Analysis of Non-Deterministic Loads

Transient and frequency response solutions are what are called deterministic loadings. That is, a particular load is described, either as a time dependent loading or as a sinusoidal loading, and the solution is found for that load. If you know the exact load that your structure is going to see, deterministic loads are the best way to look at things. However, in many cases, we don't know exactly what load is going to occur. We may know the probability of the load exceeding some limit, or we may know that the load will never exceed some limit for a given collection of loadings. Loads of this type are nondeterministic, and are commonly analyzed with one of two different types of analysis.

Nondeterministic analysis in Nastran can be broken down into two categories, random and response spectrum.

Random loads can be further subdivided:

- Stationary random loads are ones in which the mean is constant and the nature of the signal remains the same.

- Non-stationary random loads are ones in which the mean may vary and the nature of the signal changes. In this example there are 'bursts' of loading, and the characteristics of each burst are different.

An ergodic random signal is one such that a sample can be taken out of any of a number of similar event signals, or for the same time across multiple signals and it will be representative of the signal as a whole.

A random analysis can be used for loads that are stationary and ergodic. Things like earthquake ground motion, ocean wave heights and frequencies, wind pressure fluctuations on aircraft and tall buildings, and acoustic excitation due to rocket and jet engine noise are typical stationary ergodic loads.

In contrast, shock loadings or loads with large nonlinear effects may not meet this description. They are commonly non-stationary and cannot be analyzed with a random analysis. Response spectrum analysis can be used for some of these types of loads. Earthquakes and shock loadings are usually examined with response spectrum analysis.

Random Response Analysis

In a random analysis, the loading is described by a power spectral density function. Basically, this describes the mean magnitude of the loading as a function of frequency.

Consider a typical signal that meets the definition of stationary and ergodic:

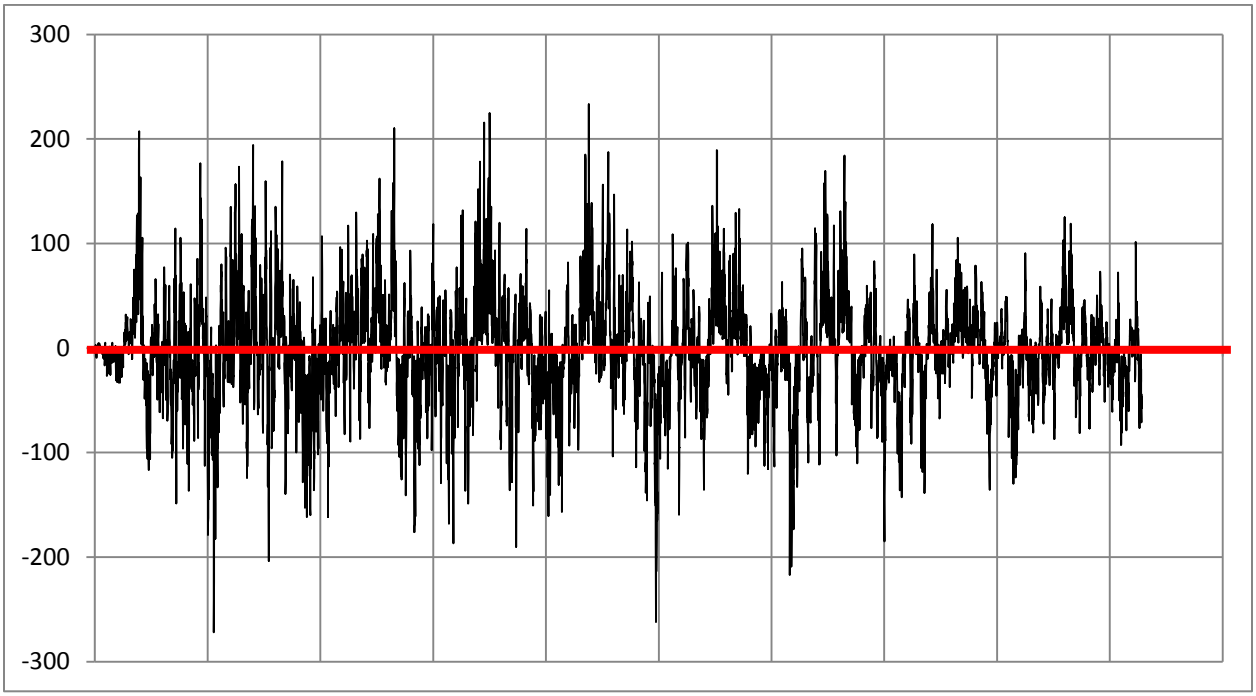

FIGURE 3

The mean of this loading is 0.0 and the signal is an acceleration measured in gs. If we square the signal at all times, we will get a signal that is now all positive and has a non-zero mean. The Y axis is G^2 now since we are squaring it:

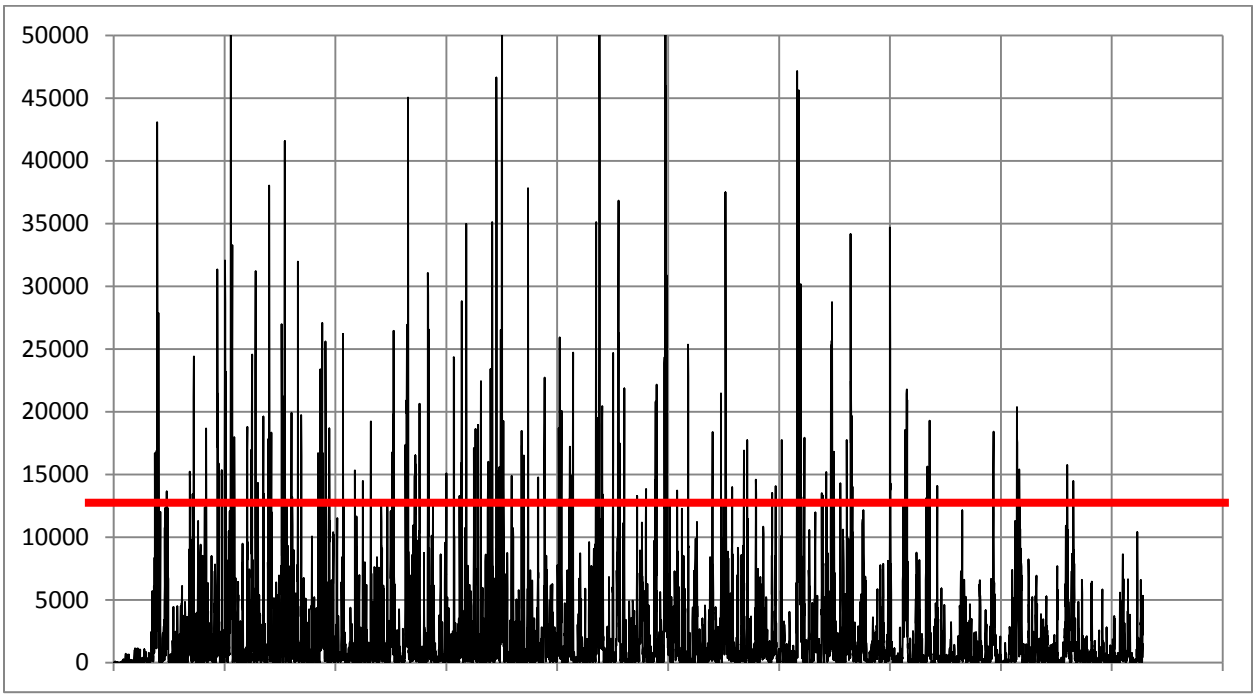

FIGURE 4

It can be shown statistically that the Square Root of the Mean Square Value (RMS) is equal to the standard deviation σ of a Normal distribution. One standard deviation σ of the RMS value of the signal is the value that has a 68.3% chance of occurring. "3σ" gives a probability of 99.73% chance of occurring. We now have a measure of the mean amplitude of the signal as its RMS value.

We can further characterize the signal. Apply a filter to the original signal so that we eliminate all frequencies above say, f1. Square the signal and find the Mean Square again. We have now recreated the same plot above, but missing some high frequency content:

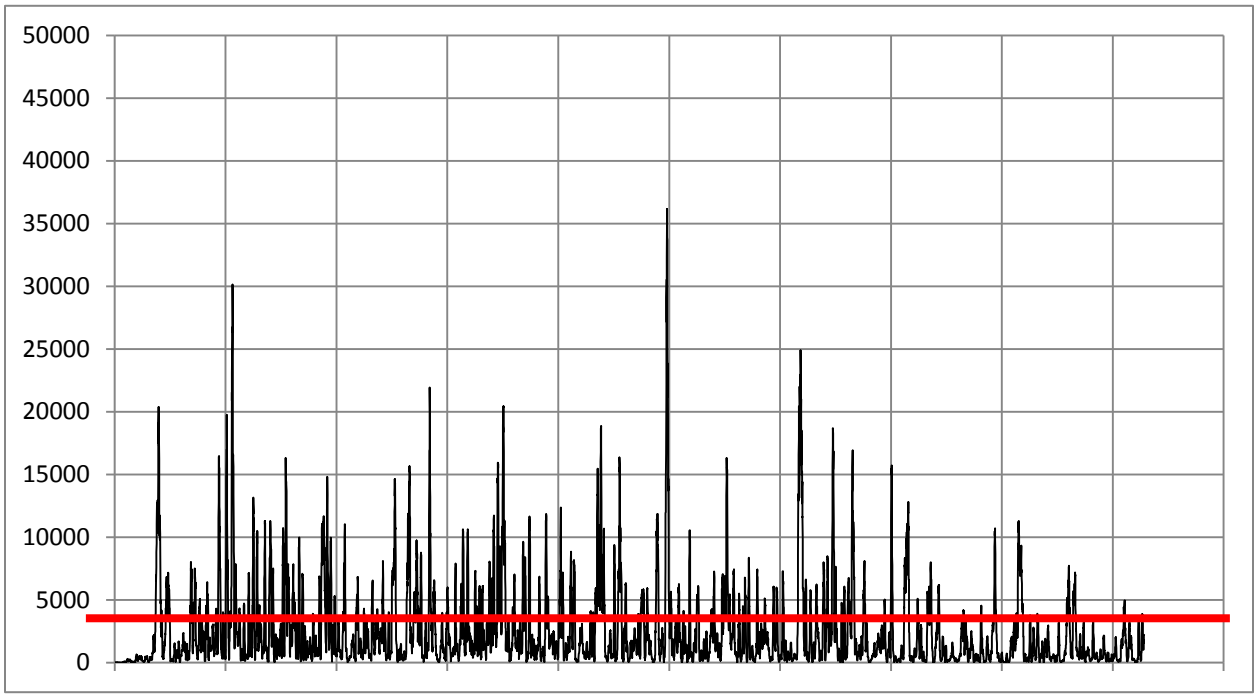

FIGURE 5

The new mean square is necessarily below the mean square of the unfiltered signal.

Continue to apply a reducing upper limit on f. As each frequency range is cut off, the Mean Square value will be decreasing.

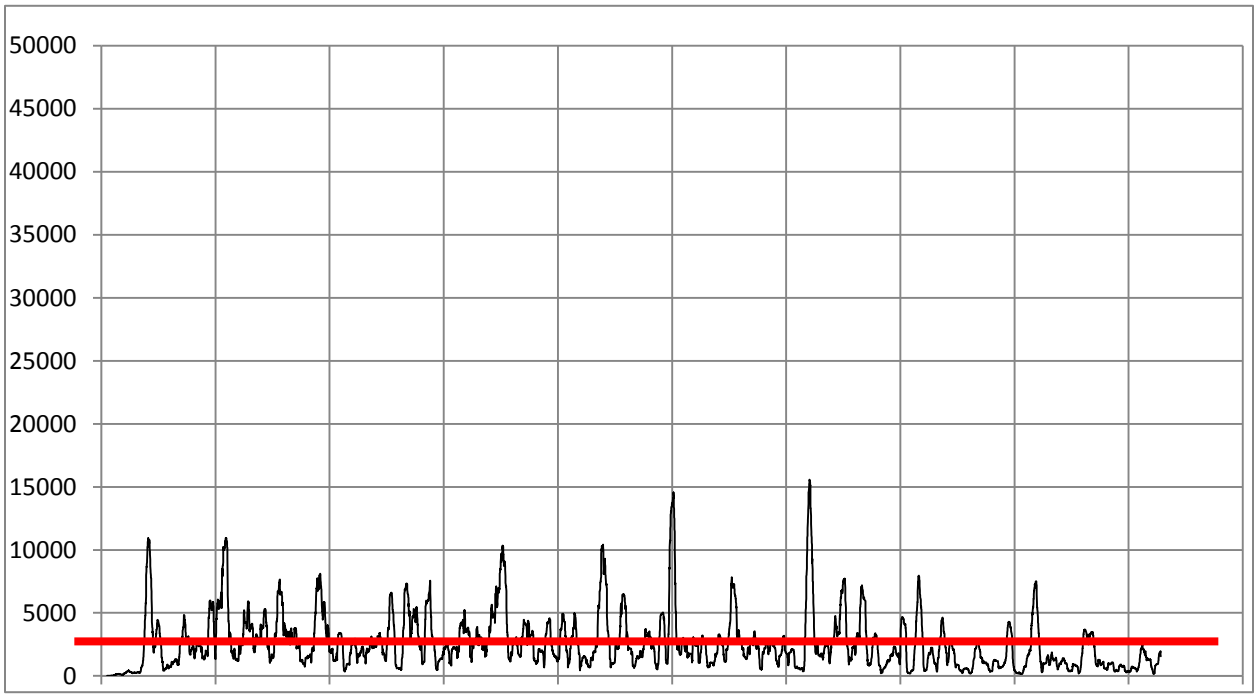

FIGURE 6

Now we can plot the variation of Mean Square with f_i.

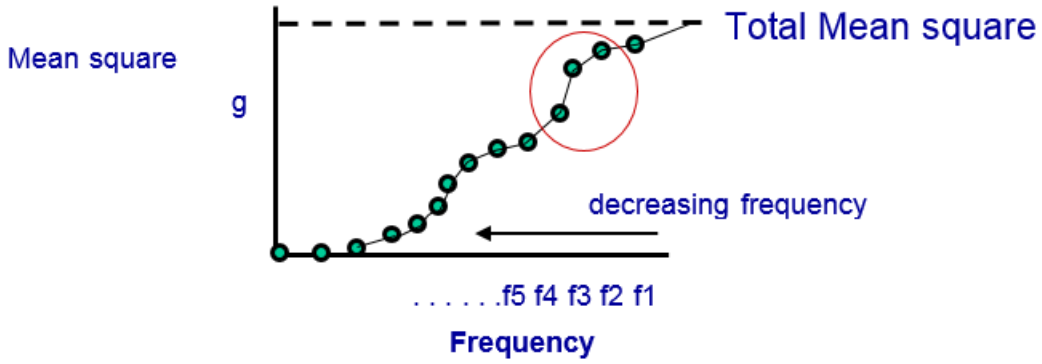

FIGURE 7

This type of plot is called the Cumulative Mean Square (CMS) plot, or if we take the square root of the terms it is the CRMS plot. It shows the frequency content of the random signal. In this example, the Mean Square value jumps considerably between f3 and f2. Note that the Total Mean Square value represents the unfiltered signal.

If we take the derivative of the CRMS plot to find the gradient, we get a plot that looks like this:

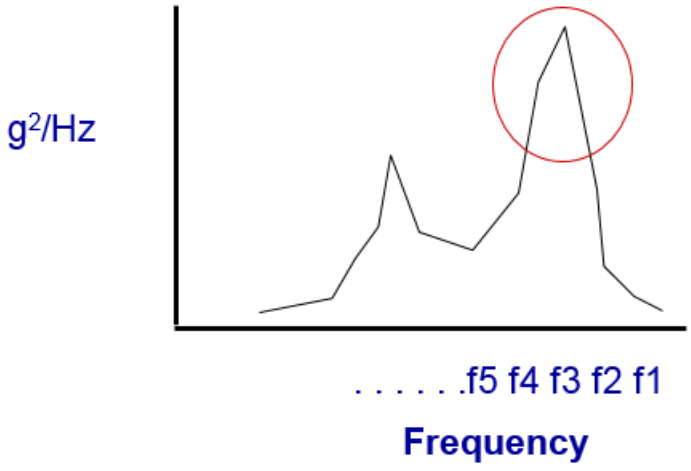

FIGURE 8

This type of plot is called the Power Spectral Density (PSD). It shows the frequency content of the random signal, more directly than the CMRS. Like the CRMS plot, the G^2/Hz value jumps considerably between f3 and f2. The square root of the area under the curve is the RMS value of the signal.

This type of plot is used as input to a random analysis, as it statistically describes the variation of the magnitude of the signal with frequency. A random analysis usually uses an acceleration vs frequency input, so it is really an ASD (acceleration spectral density).

Typically, a series of tests are run and PSD (or ASD) plots generated for each. These are then combined under an envelope such that the PSD of any of the test signals will fall under the enveloped value:

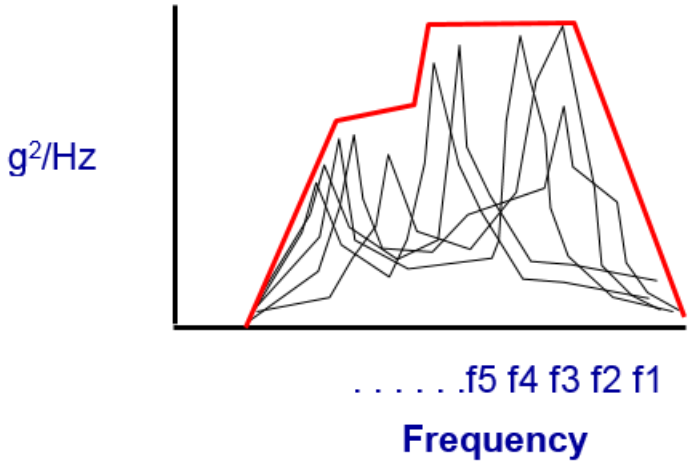

FIGURE 9

This envelope is what is commonly input to Nastran for a random analysis.

A random analysis isn't really an analysis per se. It runs as a post-process to a frequency response analysis. The model is first solved for response to a unit excitation over the range of frequencies. The resulting output is combined in different ways, and coupled with the defined PSD to generate the result of the random input excitation.

A random response model must be linear, as must the input frequency response model.

Random Output

A random analysis generates a number of outputs that are specific to random analysis. Among these are RMS values, PSD plots, number of positive crossings (NPX), cumulative RMS and Autocorrelation.

Once the frequency response analysis is run, the user has a number of pieces of data:

$H_{ia}(\omega_i)$ – frequency responses of quantities u_i to loads P_i at frequencies ω_i. The loads P_i are usually unit loads applied over a range for frequencies wi. The responses can be displacements, acceleration, force, stress, etc.

S_a – auto spectral density

S_{ab} – cross spectral densities

- The RMS values are the areas under the resultant PSD plots. We can plot as contours for stress etc. They are factored by 3 to give 3σ probability of exceedance, and the RMS values give a mean stress for fatigue analysis.

- The PSD plots show response compared to the input PSD. Important frequencies can be identified and verified with this plot.

$$S_j(\tau) = \lim_{T \to \infty} \frac{2}{T} \left| \int_0^T u e^{-\omega t} u_j(t) dt \right|^2$$

- Number of positive crossings is a statistical calculation that predicts how many zero crossings will occur per unit time of response. That is, how many times the vibration will change from negative to positive. This is also known as the apparent frequency and gives cycle count for fatigue.

- The cumulative RMS plot is another way to identify which frequencies contribute the most.

- Autocorrelation is an indication of the degree of randomness of a response. The signal is multiplied by itself with different phase shifts. If a signal is non-random (sine function, square wave etc.) then a broad correlation is seen and distinct and regular peaks will become evident. If a signal is highly random then the autocorrelation output is very 'peaky' and large irregular peaks will occur randomly.

$$R_j(\tau) = \lim_{T \to \infty} \frac{2}{T} \int_0^T u_j(t) u_j(t-\tau) dt$$

Response Spectrum Analysis

Response spectrum analysis is another nondeterministic analysis method. Whereas a random analysis runs as a post-process to a frequency response analysis, response spectrum analysis runs following a modal analysis.

A response spectrum is defined as a plot of the peak responses of a set of independently vibrating bodies to a transient input. This can be illustrated using a special model that consists of a rigid element that holds a number of small vibrating bodies, each tuned to a different frequency.

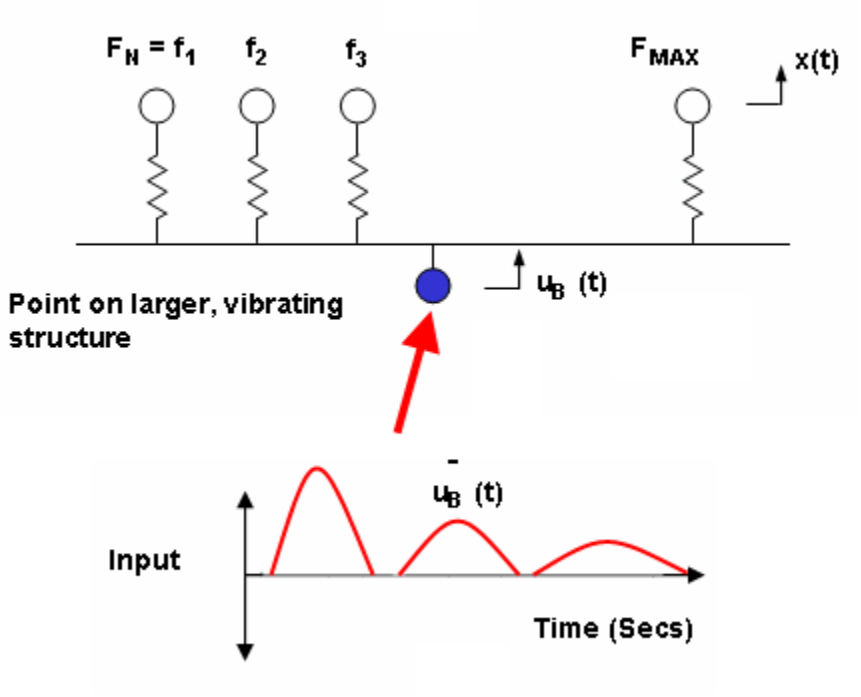

FIGURE 10

The model is subject to a transient loading of some type, and the response of each body is plotted. The peak response of each body is plotted against its frequency.

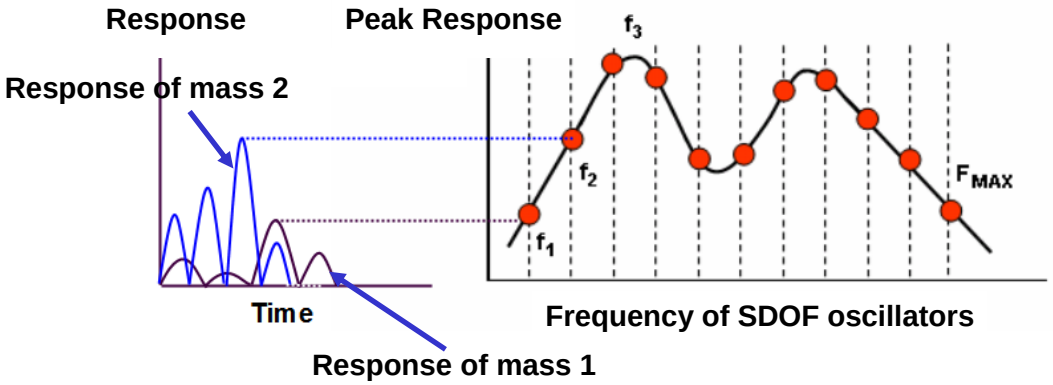

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FIGURE 11

The resulting plot of peak response vs frequency is a shock spectrum.

Response spectrum analysis can be divided into two separate and generally unrelated categories. The first concerns the creation of a spectrum from a transient analysis. The second part is the application of a spectrum to calculate results. The first is used to evaluate an environment and to potentially create a design spectrum. The second is used to evaluate a structure to a transient shock environment where the exact transient is unknown.

To create a spectrum from a multi-DOF model, the model is subjected to a transient. For each point that is listed for data recovery, a series of small oscillators are included at that point. The peak responses of the oscillators are recorded and used to generate the shock spectrum for that point. The generation of s spectrum is a deterministic analysis in that the spectrum is generated for the particular input transient. However, note that different transient loads may produce identical spectra.

The application of a shock spectrum on the other hand is non-deterministic analysis. In general, the design spectrum represents an envelope of potential transient loads. The shock environment has been analyzed or tested with many different types of loads and spectra generated from each. The design spectrum then represents the peak of these, in that no particular transient load will be expected to exceed the allowable if the part is acceptable to the response spectrum. Like a random analysis, response spectrum analysis is run as a post-process to another run. In this case, it is a modal analysis rather than a frequency response analysis.

Using a Spectrum

In many cases, what people describe as spectral response problems are really referring to the use of a design spectrum instead of the creation of a spectrum. A design spectrum can be created by running multiple spectrum generation runs from a variety of transients and then creating a spectrum from them that encompasses all the peaks. For instance, on our run, we might create a design spectrum like this:

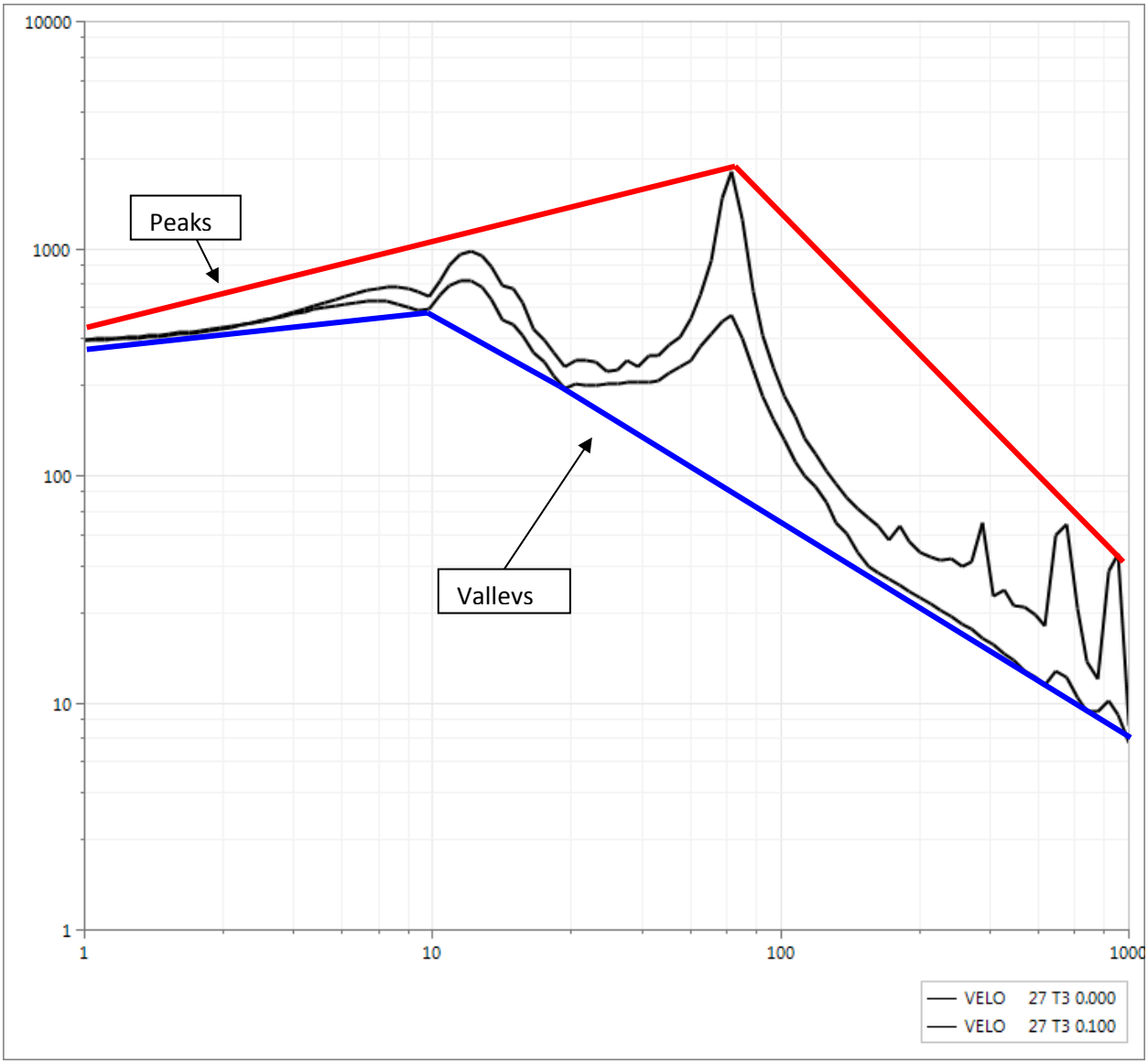

FIGURE 12

With only one transient and one model to compare, this is not necessarily a good design spectrum we have created. Often times spectra are designed such that local model resonances are removed, essentially taking the valleys of the spectrum instead of the peaks. In that case,

the second one above might be a better representation of the shock environment. The reasoning behind the valley approach is that the peaks are artifacts of model resonance. Incorporating the resonance into the design spectrum will over-excite models with the same resonances.

Using a spectrum involves converting a multi-degree-of-freedom model into a set of independent oscillators, then exciting them with the values from the spectrum. Fortunately for us, mode shapes are ideal for this purpose. They are orthogonal, and thus independent, and represent the vibrating shape of the structure. However, all modes will not be excited equally by a particular shock event, even if the spectrum is flat. For example, consider a simple cantilever beam. A shock perpendicular to the beam axis will excite the modes in that plane, but will not excite the modes in the axial direction, nor those in the plane at 90 degrees to the excitation plane.

To account for the relative ability of different modes to be excited, we can think of rigid body motion in a particular direction as being made up as a superposition of all the modes of the model. But obviously, some modes will contribute to this rigid body summation more than others. In our example above, the axial mode of the beam will not contribute to a rigid body vector in the perpendicular direction at all. Mathematically, we can describe the rigid body vector R thusly:

$$D_R = \sum \phi_i \varepsilon_i$$

The □ terms are the eigenvectors, and the □ terms are scale factors. That is, each mode will be scaled by an appropriate factor to yield the total summation. Note that this property is forced by the concept of orthogonality and having a complete set of modes – i.e. one for each degree of freedom.

In matrix terms,

$$\{D_R\} = [\phi]\{\varepsilon\}$$

If we pre-multiply each side by $[\square]^T[M]$, we get

$$[\phi]^T[M]\{D_R\} = [\phi]^T[M][\phi]\{\varepsilon\}$$

We recall that $[\square]^T[M][\square]$ is the generalized mass $[M_{jj}]$, a diagonal matrix with one term for each mode in the model. Also, note that for a mass scaled eigenvectors, this term is an identity matrix, and that the rigid body vector is simply a column of 1s. Thus we can show that the scale factor is a simple product based on the mass and the eigenvectors

$$[\phi]^T [M]\{1\} = \{\varepsilon\}$$

This scale factor is commonly called the participation factor P, and is a second scale factor on the response in a response spectrum analysis. That is, each mode shape is scaled by two factors to come up with a model response to a particular mode. For example, the response acceleration at node i for mode j will be found from the participation factor, the spectrum acceleration for that frequency, and the eigenvector at that node for that frequency.

$$a_{ij} = P_{ij} A_j \phi_{ij}$$

Because this is a linear model, displacements, velocities and accelerations are directly related by the natural frequency. The displacement at the same node would be found from the following:

$$u_{ij} = \frac{P_{ij} A_j \phi_{ij}}{\omega^2}$$

The element force and stress results are then found from the displacements for each mode.

Recall that we had indicated that the scale factors for certain modes might be small or zero for some modes in some directions. We can solve for the scale factors determine if they are significant. However, there is a somewhat easier way available to us, using the concept of modal effective mass. We can demonstrate that the modal effective mass for any mode is proportional to the scale factor, and that the sum of all the modal masses will be equal the total mass of the system. Thus, the magnitude of a modal mass can be easily quantified and used to determine the relative importance of any mode. That is, a large modal mass as a percentage of the total model mass will be important and have a large scale factor. A small modal mass indicates a small participation factor, and little contribution to the response.

Let's look at what we exactly mean by modal effective mass. We can apply similar logic to the mass that is moving in each direction as we did to the rigid body vector. Define a 'rigid body mass" that is defined as the sum of individual generalized masses contributed by each degree of freedom

$$\{M_R\} = \{D_R\}^T [M]\{D_R\}$$

Because of orthogonality, we know that the sum of the terms of M_R is the total mass of the system. And from earlier, the rigid body vector D_R = [□]{□}, so that

$$\{M_R\} = [\phi]^T \{\varepsilon\}^T [M][\phi]\{\varepsilon\}$$

And seeing the generalized mass term again, we can substitute:

$$\{M_R\} = \{\varepsilon\}^T [M_{jj}]\{\varepsilon\}$$

If the Eigenvectors are mass scaled, $[M_{jj}] = 1$, so we end up with

$$\{M_R\} = \{\varepsilon^2\}$$

In other words, the modal effective mass, or the mass contributed by each mode to the motion in a particular direction is simply the square of the scale factors for each mode.

Thus by looking at the modal mass, one can easily determine the contribution of that mode to the overall response.

Recall that a response spectrum analysis will produce a response for each mode that is effectively a static analysis. To represent what really happens in a shock event, it would be necessary to take the scaled response for each mode and write it out in time as a sinusoidal response. Then, the different modes would be added up at each frequency, accounting for the different frequencies and phasing of the different modes. This is a complete quadratic combination, and effectively rebuilds the transient response from the individual modal responses, converting the model from the frequency domain back into the time domain.

In practice, this is occasionally done. However, the approach is usually simplified, and a rough combination of the peaks is made. Common methods to do this are the SRSS (square root of the sum of the squares) and NRL (a Naval Research Lab modification of the SRSS method). In those combinations, only the peak responses are considered for each mode, and the phasing caused by the different frequencies is ignored. The resulting displacement of a node, for example, using the SRSS summation will be:

$$u_{SRSS} = \sqrt{\sum u_i^{\,2}}$$

u_i is the response of that node from each mode i. In theory, if time were run out long enough, eventually all the modes would combine constructively, resulting in a peak response that was

the absolute sum of the individual responses. However, response spectrum analysis is often used for transient shock events in systems with damping. In that situation, the responses of the individual modes will die out over time, with the higher frequencies dying out faster. As a result, as time progresses, the sums will generally get smaller and smaller. Thus a summation that takes an RMS sum of the peaks will usually suffice to represent the maximum responses that would ever be seen.

Another concept in the summation is that modes with small scale factors or small modal effective masses will contribute very little to the net response. As a result, it is often possible to create a summation from a smaller number of important modes that is almost identical to a full summation of all modes. Most SRSS techniques exploit this by allowing the extraction of a limited number of modes and summing only the contributions of those relatively low frequency modes. The technique can be further improved by limiting the summation to only to modes with a significant contribution of modal mass. Thus is it often possible to obtain almost identical results from a run with a small number of modes and that with a large number, provided all of the important modes have been included in the smaller case.

Dynamic Reduction

Component Mode Synthesis

Component mode synthesis is a method for replacing a many-degree-of-freedom portion of a large modal with a much smaller modal representation. The advantage of this process is that a much smaller model can be run. The primary disadvantage is that once the model is reduced to its compact modal representation, it cannot be expanded back to its full form.

CMS is run it two steps. In the first step, the component model is run for modes and reduced to a matrix. The matrix is stored in a file as a DMIG entry. In the second step, the reduced component is included in the rest of the model and run.

A Crag-Bampton reduction starts with the equations of motion.

$$[M]\ddot{x}(t) + [B]\dot{x}(t) + [K]x(t) = \{P(\omega)\}e^{i\omega t}$$

We will define a transformation, or partitioning of the displacement matrix into “Interior” DOF and “Boundary” DOF.

$$\{u_A\} = \begin{Bmatrix} u_b \\ \hline u_L \end{Bmatrix} = \begin{bmatrix} I & 0 \\ \phi_R & \phi_L \end{bmatrix} \begin{Bmatrix} u_b \\ q \end{Bmatrix} \qquad (2)$$

This partitioned matrix is built from three easily accessible matrices; An identity matrix I, the fixed base mode shapes ϕ_L and a rigid body vector ϕ_R. As a result of this transformation, the displacements are broken into the displacements of the boundary nodes, u_b and the displacements of 'modal' degrees of freedom q. The matrix of vectors is the Craig-Bampton transformation matrix, $[\phi_{cb}]$

This transformed relation can be substituted back into the equations of motion, then pre-multiplied by $[\phi_{cb}]^T$ to arrive at:

$$\phi_{cb}^T [M_{AA}] \phi_{cb} \begin{Bmatrix} \ddot{u}_b \\ \ddot{q} \end{Bmatrix} + \phi_{cb}^T [K_{AA}] \phi_{cb} \begin{Bmatrix} u_b \\ q \end{Bmatrix} = \phi_{cb}^T \begin{Bmatrix} F_b \\ F_L \end{Bmatrix} \qquad (3)$$

This equation contains two terms that look oddly familiar, a mass and a stiffness matrix that are defined by $[\phi]^T[M][\phi]$ and $[\phi]^T[K][\phi]$, like the generalized mass and stiffness matrices. In this transformation, however, these are defined slightly differently in terms of transformed coordinates

$$[M_{cb}] = \phi_{cb}^T [M_{AA}] \phi_{cb} = \begin{bmatrix} M_{bb} & M_{bq} \\ M_{qb} & M_{qq} \end{bmatrix}$$

$$[K_{cb}] = \phi_{cb}^T [K_{AA}] \phi_{cb} = \begin{bmatrix} K_{bb} & 0 \\ 0 & K_{qq} \end{bmatrix}$$

If externally applied forces are limited to boundary points, this can be rewritten as

$$\begin{bmatrix} M_{bb} & M_{bq} \\ M_{qb} & M_{qq} \end{bmatrix} \begin{Bmatrix} \ddot{u}_b \\ \ddot{q} \end{Bmatrix} + \begin{bmatrix} K_{bb} & 0 \\ 0 & K_{qq} \end{bmatrix} \begin{Bmatrix} u_b \\ q \end{Bmatrix} = \begin{Bmatrix} F_b \\ 0 \end{Bmatrix}$$

The equations of motion are now described wholly in terms of things we know. K_{qq} will be the generalized stiffness associated with the individual modes. M_{qq} are the generalized masses associated with the modes. M_{bb} is the mass of the model transformed to the boundary points, and K_{bb} is the same. If modes are mass normalized, M_{qq} is an identity matrix, and K_{qq} is the matrix of ω^2.

Damping can then be included directly as modal damping, with a term for each frequency.

$$\begin{bmatrix} M_{bb} & M_{bq} \\ M_{qb} & I \end{bmatrix} \begin{Bmatrix} \ddot{u}_b \\ \ddot{q} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2\zeta\omega \end{bmatrix} \begin{Bmatrix} \dot{u}_b \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} K_{bb} & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{Bmatrix} u_b \\ q \end{Bmatrix} = \begin{Bmatrix} F_b \\ 0 \end{Bmatrix}$$

The advantage of this transformation procedure is that instead of a full set of degrees of freedom, we now have b+q DOF, or the number of boundary DOF + the number of modal DOF. In this way, the full dynamic behavior of a model can be reduced to a much more limited set of DOF. Further, it can be shown that DOF with very small modal mass will contribute little or nothing to the solution, and thus only modes with large participation factors or modal masses need be included in the transformation.